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Some Convolution Identities for Catalan Numbers

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Abstract

In the paper we propose a series of convolution identities for the sequence of Catalan numbers. The proofs of two of them are provided and the other identities are proposed as conjectures. We include a simple Maple code to generate and confirm the identities.

Keywords: Catalan number; convolution identity; binomial.

1 Introduction

Probably the most prominent among the special integers that arise in combinatorial contexts are the binomial coefficients $\binom{n}{m}$. In this paper we discuss an important sequence called Catalan numbers. The sequence of Catalan numbers C_n was described in the 18th century by Leonhard Euler. The sequence is named after the Belgian mathematician Eug'ene Charles Catalan.

The nth Catalan number is given directly in terms of binomial coefficients as follows,

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!} \qquad \text{for } n \geq 0. \tag{1}$$

The first Catalan numbers for $n = 0, 1, 2, 3, 4, 5, \ldots$ are

$$1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429, 1430, 4862, 16796, 58786, \ldots$$

For detailed information, please refer to the monograph [2].

A recursive definition of C_n have been given by L.Euler in 1761 as follows,

$$C_0 = 1, \quad C_n = \frac{4n-2}{n+1} C_{n-1}, \;\; n \geq 1.$$

Although Euler and Catalan are recognised for discovering the Catalan numbers' sequence, some believe that a Chinese mathematician named Antu Ming (1692–1763) was the pioneer. In the 1730's, he focused on the geometric meaning of the Catalan number. He worked at the Qing court in China as an astronomer, mathematician, and topographic scientist. He is called Minggatu (full name Sharabiin Myangat), Ming Antu (Chinese name) and Jing An (courtesy name). Ming Antu wrote a book Quick Methods for Accurate Values of Circle Segments, which contained several trigonometric identities and power series, some concerning Catalan numbers.

Ming Antu also obtained the recurrence formula

$$C_1 = 1, \quad C_2 = 2, \quad C_{n+1} = \sum_{s=0}^{n} (-1)^s \binom{n+1-s}{s+1} C_{n-s}.$$

He seems to have no hint of a combinatorial interpretation of Catalan numbers.

Ming Antu's book was published only in 1839, and the connection to Catalan numbers was observed by Luo Jianjin in 1988 (see [3, 4] and for further information also see [11] (Appendix B)).

Eugene Charles Catalan (1814–1894) "rediscovered" the Catalan Numbers while exploring well-formed sequences of parentheses in 1838. He was a Belgian mathematician who stated the famous Catalan conjecture. Although this set of numbers is named after him, he was not the first to discover these numbers.

The Catalan sequence also was worked out by Leonhard Euler(1707-1783), who was interested in the number of different ways of dividing a polygon into triangles. L. Euler reviewed this number set related to the triangulations of convex polygons. He established a recursive formula in 1761 and collaborated with Hungarian mathematician, Johann von Segner (1704-1777), to derive a second order recurrence relation.

Gabriel Lame (1795-1870), was a French mathematician who used Euler's recursive formulae to find an explicit formula in 1838.

In spite of a numerous generalizations and recursive formulae, books written, papers published and applications given the sequence of Catalan numbers is still of scientists' great interest (see [6, 7, 12] and others).

In this paper we propose a series of convolution identities for the sequence of Catalan numbers. The identities came out from an intension to create isomorphism criteria for some classes of Leibniz algebras. The identities have been used to create such isomorphism criteria for low-dimensional filiform Leibniz algebras in [9] ($t = 5, 6$ and $t = 7$ cases), [10] ($t = 8$) and [5] ($t = 9, 10$). All these cases the corresponding identities were proven by computations. First time in the most general form the identities were exhibited in [8]. Since then we received a lot of requests to provide the proofs of the identities. In fact, a hint was given in [8] to use a combination of properties of the binomials and Catalan numbers. To prove the identities for $k = 1$ and $k = 2$ we used the so-called "halving trick".

The organization of the paper is as follows. In Section 2 we recall some statements and identities on the binomial coefficients and Catalan numbers. The convolution identities for Catalan numbers are exhibited in Section 3. In APPENDIX section of the paper we include a simple Maple code to generate and verify the identities proposed for integers $t \geq 3$ and $1 \leq k \leq t-3$.

2 Some auxiliary identities

The purpose of this section is to recall some identities on the sequence of Catalan Numbers and Binomial Coefficients (without claiming any originality) to use them later, they can be found in the literature (see, for example, [1, 2, 11]). These are included here in order to make the paper a self-contained.

The definition of the binomial coefficients are given as follows

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{n(n-1)(n-2)\dots(n-(m-1))}{m!} = \frac{n(n-1)(n-2)\dots(m+1)}{(n-m)!}. \tag{2}$$

The definition (2.1) supposes that n and m to be positive, but due to numerous applications of $\binom{n}{m}$ in mathematics and beyond one has to define it for negative numbers and fractional numbers as well.

Binomial Coefficients for a factional number $\frac{p}{q}$ as an upper index and an integer m as a lower index are defined as follows

$$\binom{\frac{p}{q}}{m} = \frac{\frac{p}{q}(\frac{p}{q}-1)(\frac{p}{q}-2)\dots(\frac{p}{q}-(m-1))}{m!}.$$

Proposition 2.1. *The following relation between "halving" and Catalan numbers holds true*

$$\binom{\frac{1}{2}}{s} = (-1)^{s-1}\frac{2}{4^s}C_{s-1}.$$

Proof.

$$\begin{array}{rcl}\binom{\frac{1}{2}}{s} & = & \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-(s-1))}{s!}\\ & = & \frac{(-1)^{s-1}\cdot 1\cdot(2\cdot 1-1)\cdot(2\cdot 2-1)\cdot\ldots\cdot[(2\cdot(s-1)-1)}{2^s s!}\\ & = & (-1)^{s-1}\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2s-3)}{2^s s!}\\ & = & (-1)^{s-1}\frac{(2s-2)!}{2^s s![2\cdot 4\cdot 6\cdot\ldots\cdot(2s-2)]}\\ & = & (-1)^{s-1}\frac{(2s-2)!}{2^s s! 2^{s-1}(s-1)!}\\ & = & (-1)^{s-1}\frac{(2s-2)!}{2^{2s-1}s!(s-1)!}\\ & = & \frac{(-1)^{s-1}}{2^{2s-1}s}\binom{2s-2}{s-1}\\ & = & \frac{(-1)^{s-1}}{2^{2s-1}}C_{s-1}\\ & = & (-1)^{s-1}\frac{2}{4^s}C_{s-1}.\end{array}$$

□

The definition of the binomial coefficients for a negative upper index is given as follows

$$\binom{-n}{m} = \frac{-n(-n-1)(-n-2)\dots(-n-m+1)}{m!} = (-1)^m\frac{n(n+1)(n+2)\dots(n+m-1)}{m!}. \quad (3)$$

Accordingly for a factional number $-\frac{p}{q}$ as an upper index and an integer m as a lower index it becomes

$$\binom{-\frac{p}{q}}{m} = \frac{-p(-p-q)(-p-2q)\cdots[-p-(m-1)q}{q^m m!} = (-1)^m\frac{p(p+q)(p+2q)\cdots[p+(m-1)q}{q^m m!}.$$

Proposition 2.2. *One has the following identity*

$$\binom{-\frac{1}{2}}{s} = \frac{(-1)^s}{4^s}(1+s)C_s.$$

Proof. Indeed,

$$\begin{array}{rcl}\binom{-\frac{1}{2}}{s} & = & (-1)^s\frac{1\cdot(1+2\cdot 1)\cdot(1+2\cdot 2)\cdot\ldots\cdot[1+2\cdot(s-1)]}{2^s s!}\\ & = & (-1)^s\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2s-1)}{2^s s!}\\ & = & (-1)^s\frac{(2s-1)!}{2^s s![2\cdot 4\cdot 6\cdot\ldots\cdot(2s-2)]}\\ & = & (-1)^s\frac{(2s-1)!}{2^s s! 2^{s-1}(s-1)!}\\ & = & (-1)^s\frac{(2s)!}{2^{2s}s!s!}\\ & = & \frac{(-1)^s}{2^{2s}}\binom{2s}{s}\\ & = & \frac{(-1)^s}{4^s}(1+s)C_s.\end{array}$$

□

The following recurrent formula is proven by using generating function

$$C(x) = \sum_{s=0}^{\infty} C_s x^s = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots$$

for Catalan sequence $\{C_0, C_1, C_2, \ldots\}$.

Proposition 2.3. (*Segner's recurrence relation*) *For any natural number* $n = 0, 1, 2, ...$ *one has*

$$\sum_{s=0}^{n} C_s C_{n-s} = C_{n+1}. \tag{4}$$

Proof. Consider

$$C(x)^2 == 1 + (C_0C_1 + C_1C_0)x + (C_0C_2 + C_1C_1 + C_2C_0)x^2 + ... = \sum_{n=0}^{\infty}\left(\sum_{s=0}^{n} C_s C_{n-s}\right) x^s.$$

Let $A(x) = \frac{d}{dx}[xC(x)]$. Then

$$A(x) = \sum_{s=0}^{\infty}\binom{2s}{s}x^s = \frac{1}{\sqrt{1-4x}}.$$

Therefore,

$$xC(x) = \int_0^x A(t)dt = \int_0^x \frac{1}{\sqrt{1-4t}}dt = \frac{1}{2}(1 - \sqrt{1-4x}.$$

Hence,

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

and

$$C(x)^2 = \frac{1-\sqrt{1-4x}}{2x^2} - \frac{1}{x} = \frac{C(x)-1}{x} = \frac{1}{x}\sum_{s=0}^{\infty} C_s x^s = \sum_{s=0}^{\infty} C_{s+1}x^s.$$

Thus,

$$C_{n+1} = \sum_{s=0}^{n} C_s C_{n-s}.$$

□

Proposition 2.4. *For the sequence of Catalan numbers the following identity takes place*

$$\sum_{s=0}^{n} sC_sC_{n-s} = \frac{n}{2}C_{n+1}, \text{ where } n = 0, 1, 2, ... \tag{5}$$

Proof.

$$\sum_{s=0}^{n} sC_sC_{n-s} = \sum_{r=0}^{n}(n-r)C_rC_{n-r} = n\sum_{r=0}^{n} C_rC_{n-r} - \sum_{r=0}^{n} rC_rC_{n-r}.$$

Therefore,

$$2\sum_{s=0}^{n} sC_sC_{n-s} = n\sum_{r=0}^{n} C_rC_{n-r} = nC_{n+1}.$$

Thus,

$$\sum_{s=0}^{n} sC_sC_{n-s} = \frac{n}{2}C_{n+1}.$$

□

We also make use the following well-known Chu–Vandermonde Convolution formula (see [2]):

Proposition 2.5. *For rational α, β and a positive integer n the following holds true*

$$\sum_{s=0}^{n}\binom{\alpha}{s}\binom{\beta}{n-s}=\binom{\alpha+\beta}{n}.$$

3 The identities

In this section we set up two convolution identities for Catalan numbers and give proofs of them.

Theorem 3.1. *For Catalan numbers and for a natural number t, $(t \geq 3)$ the following identities hold true*

a) $A(1) := \sum\limits_{s=1}^{t-3}\left(\begin{array}{c} s+1 \\ s \end{array}\right) C_s C_{F(1)} = \left(\begin{array}{c} t-3 \\ 1 \end{array}\right) C_{t-2};$

b) $A(2) := \sum\limits_{s=1}^{t-3}\binom{s+1}{s}\binom{s-1}{1} C_s C_{t-s-2} - \sum\limits_{s=1}^{t-3}\binom{s+1}{s-1} C_s \sum\limits_{p=1}^{t-s-3} C_p C_{t-p-s-2} = \binom{t-3}{2} C_{t-2}.$

Proof. • Part a). We give two proofs of the identity. The first proving is based on the classical identity (4) for Catalan numbers along with the identity (5) and the second one is to demonstrate the proving technique applied in the proof of the convolution identity given in Part b).

$$\begin{aligned}
* \ A(1) &:= \sum_{s=1}^{t-3}\left(\begin{array}{c} s+1 \\ s \end{array}\right) C_s C_{t-s-2} = \sum_{s=0}^{t-2}(s+1) C_s C_{t-s-2} - C_{t-2} - (t-1) C_{t-2} \\
&= \sum_{s=0}^{t-2} s C_s C_{t-s-2} + \sum_{s=0}^{t-2} C_s C_{t-s-2} - t C_{t-2} = \tfrac{t-2}{2} C_{t-1} + C_{t-1} - t C_{t-2} = t C_{t-1} - t C_{t-2} \\
&= t \tfrac{2(2t-3)}{t} C_{t-2} - t C_{t-2} = (t-3) C_{t-2} = \tbinom{t-3}{1} C_{t-2}.
\end{aligned}$$

$$\begin{aligned}
* \ A(1) &:= \sum_{s=1}^{t-3}\left(\begin{array}{c} s+1 \\ s \end{array}\right) C_s C_{F(1)} = \sum_{s=1}^{t-3}(s+1) C_s C_{t-s-2} \\
&= \sum_{s=1}^{t-3}(-1)^s 4^s \tbinom{-\frac{1}{2}}{s}(-1)^{t-s-2} 2^{2(t-1-s)-1}\tbinom{\frac{1}{2}}{t-s-1} \\
&= (-1)^{t-2} 2^{2t-3}\left(\sum_{s=0}^{t-1}\tbinom{-\frac{1}{2}}{s}\tbinom{\frac{1}{2}}{t-s-1} - \tbinom{-\frac{1}{2}}{0}\tbinom{\frac{1}{2}}{t-1} - \tbinom{-\frac{1}{2}}{t-2}\tbinom{\frac{1}{2}}{1} - \tbinom{-\frac{1}{2}}{t-1}\tbinom{\frac{1}{2}}{0}\right) \\
&= (-1)^{t-2}\tfrac{4^t}{8}\left(-\tfrac{(-1)^{t-2} 8}{4^t} C_{t-2} - \tfrac{(-1)^{t-2} 8(t-1)}{4^t} C_{t-2} + \tfrac{(-1)^{t-2} 4t}{4^t} C_{t-1}\right) \\
&= -t C_{t-2} + (2t-3) C_{t-2} = (t-3) C_{t-2} = \left(\begin{array}{c} t-3 \\ 1 \end{array}\right) C_{t-2}.
\end{aligned}$$

• Part b). $A(2) := \sum\limits_{s=1}^{t-3}\binom{s+1}{s}\binom{s-1}{1} C_s C_{t-s-2} - \sum\limits_{s=1}^{t-3}\binom{s+1}{s-1} C_s \sum\limits_{p=1}^{t-s-3} C_p C_{t-p-s-2}$

$$= \sum_{s=1}^{t-3}(s+1)(s-1) C_s C_{t-s-2} - \frac{1}{2}\sum_{s=1}^{t-3}(s+1) s C_s (C_{t-s-1} - 2C_{t-s-2}).$$

First we make use of Propositions 2.1 to obtain

$$A(2) := \sum_{s=1}^{t-3}(-1)^s 4^s \binom{-\frac{1}{2}}{s}\left[(t-2)C_{t-s-2}-(t-s-1)C_{t-s-2}\right]$$
$$-\frac{1}{2}\sum_{s=1}^{t-3}(-1)^s 4^s \binom{-\frac{1}{2}}{s}\left[tC_{t-s-1}-(t-s)C_{t-s-1}\right]$$
$$+\sum_{s=1}^{t-3}(-1)^s 4^s \binom{-\frac{1}{2}}{s}\left[(t-1)C_{t-s-2}-(t-s-1)C_{t-s-2}\right].$$

Then due to Proposition 2.2 continuing we get

$$A(2) := \sum_{s=1}^{t-3}(-1)^s 4^s \binom{-\frac{1}{2}}{s}(t-2)\frac{4^{t-s-1}}{2}(-1)^{t-s-2}\binom{\frac{1}{2}}{t-s-1}$$
$$-\sum_{s=1}^{t-3}(-1)^s 4^s \binom{-\frac{1}{2}}{s}(-1)^{t-s-2}4^{t-s-2}\binom{-\frac{1}{2}}{t-s-2}$$
$$-\frac{1}{2}\sum_{s=1}^{t-3}(-1)^s 4^s \binom{-\frac{1}{2}}{s}t\frac{4^{t-s}}{2}(-1)^{t-s-1}\binom{\frac{1}{2}}{t-s}+\frac{1}{2}\sum_{s=1}^{t-3}(-1)^s 4^s \binom{-\frac{1}{2}}{s}(-1)^{t-s-1}4^{t-s-1}\binom{-\frac{1}{2}}{t-s-1}$$
$$+\sum_{s=1}^{t-3}(-1)^s 4^s \binom{-\frac{1}{2}}{s}(t-1)\frac{4^{t-s-1}}{2}(-1)^{t-s-2}\binom{\frac{1}{2}}{t-s-1}-\sum_{s=1}^{t-3}(-1)^s 4^s \binom{-\frac{1}{2}}{s}(-1)^{t-s-2}4^{t-s-2}\binom{-\frac{1}{2}}{t-s-2}$$
$$=\frac{4^{t-1}}{2}(-1)^{t-2}(t-2)\sum_{s=1}^{t-3}\binom{-\frac{1}{2}}{s}\binom{\frac{1}{2}}{t-s-1}-4^{t-2}(-1)^{t-2}\sum_{s=1}^{t-3}\binom{-\frac{1}{2}}{s}\binom{-\frac{1}{2}}{t-s-2}$$
$$-4^{t-1}(-1)^{t-1}t\sum_{s=1}^{t-3}\binom{-\frac{1}{2}}{s}\binom{\frac{1}{2}}{t-s}+\frac{4^{t-1}}{2}(-1)^{t-1}\sum_{s=1}^{t-3}\binom{-\frac{1}{2}}{s}\binom{-\frac{1}{2}}{t-s-1}$$
$$+\frac{4^{t-1}}{2}(-1)^{t-2}(t-1)\sum_{s=1}^{t-3}\binom{-\frac{1}{2}}{s}\binom{\frac{1}{2}}{t-s-1}-4^{t-2}(-1)^{t-2}\sum_{s=1}^{t-3}\binom{-\frac{1}{2}}{s}\binom{-\frac{1}{2}}{t-s-2}$$
$$=\frac{4^{t-1}}{2}(-1)^{t-2}(t-2)\left[\sum_{s=0}^{t-1}\binom{-\frac{1}{2}}{s}\binom{\frac{1}{2}}{t-s-1}-\binom{-\frac{1}{2}}{0}\binom{\frac{1}{2}}{t-1}-\binom{-\frac{1}{2}}{t-2}\binom{\frac{1}{2}}{1}-\binom{-\frac{1}{2}}{t-1}\binom{\frac{1}{2}}{0}\right]$$
$$-4^{t-2}(-1)^{t-2}\left[\sum_{s=0}^{t-2}\binom{-\frac{1}{2}}{s}\binom{-\frac{1}{2}}{t-s-2}-\binom{-\frac{1}{2}}{0}\binom{-\frac{1}{2}}{t-2}-\binom{-\frac{1}{2}}{t-2}\binom{-\frac{1}{2}}{0}\right]$$
$$-4^{t-1}(-1)^{t-1}t\left[\sum_{s=0}^{t}\binom{-\frac{1}{2}}{s}\binom{\frac{1}{2}}{t-s}-\binom{-\frac{1}{2}}{0}\binom{\frac{1}{2}}{t}-\binom{-\frac{1}{2}}{t-2}\binom{\frac{1}{2}}{2}-\binom{-\frac{1}{2}}{t-1}\binom{\frac{1}{2}}{1}-\binom{-\frac{1}{2}}{t}\binom{\frac{1}{2}}{0}\right]$$
$$+\frac{4^{t-1}}{2}(-1)^{t-1}\left[\sum_{s=0}^{t-1}\binom{-\frac{1}{2}}{s}\binom{-\frac{1}{2}}{t-s-1}-\binom{-\frac{1}{2}}{0}\binom{-\frac{1}{2}}{t-1}-\binom{-\frac{1}{2}}{t-2}\binom{-\frac{1}{2}}{1}-\binom{-\frac{1}{2}}{t-1}\binom{-\frac{1}{2}}{0}\right]$$
$$+\frac{4^{t-1}}{2}(-1)^{t-2}(t-1)\left[\sum_{s=0}^{t-1}\binom{-\frac{1}{2}}{s}\binom{\frac{1}{2}}{t-s-1}-\binom{-\frac{1}{2}}{0}\binom{\frac{1}{2}}{t-1}-\binom{-\frac{1}{2}}{t-2}\binom{\frac{1}{2}}{1}-\binom{-\frac{1}{2}}{t-1}\binom{\frac{1}{2}}{0}\right]$$
$$-4^{t-2}(-1)^{t-2}\left[\sum_{s=0}^{t-2}\binom{-\frac{1}{2}}{s}\binom{-\frac{1}{2}}{t-s-2}-\binom{-\frac{1}{2}}{0}\binom{-\frac{1}{2}}{t-2}-\binom{-\frac{1}{2}}{t-2}\binom{-\frac{1}{2}}{0}\right].$$

Now we apply the Chu–Vandermonde Convolution formula (Proposition 2.5) and get

$$A(2) := \frac{4^{t-1}}{2}(-1)^{t-2}(t-2)\left[-\binom{\frac{1}{2}}{t-1}-\binom{-\frac{1}{2}}{t-2}\frac{1}{2}-\binom{-\frac{1}{2}}{t-1}\right]-4^{t-2}(-1)^{t-2}\left[(-1)^{t-2}-2\binom{-\frac{1}{2}}{t-2}\right]$$
$$-4^{t-1}(-1)^{t-1}t\left[-\binom{\frac{1}{2}}{t}-\binom{-\frac{1}{2}}{t-2}\left(-\frac{1}{8}\right)-\binom{-\frac{1}{2}}{t-1}\frac{1}{2}-\binom{-\frac{1}{2}}{t}\right]$$
$$+\frac{4^{t-1}}{2}(-1)^{t-1}\left[(-1)^{t-1}-\binom{-\frac{1}{2}}{t-1}-\binom{-\frac{1}{2}}{t-2}\left(-\frac{1}{2}\right)-\binom{-\frac{1}{2}}{t-1}\right]$$
$$+\frac{4^{t-1}}{2}(-1)^{t-2}(t-1)\left[-\binom{\frac{1}{2}}{t-1}-\binom{-\frac{1}{2}}{t-2}\frac{1}{2}-\binom{-\frac{1}{2}}{t-1}\right]-4^{t-2}(-1)^{t-2}\left[(-1)^{t-2}-2\binom{-\frac{1}{2}}{t-2}\right].$$

Finally, converting the binomials into Catalan numbers by using Propositions 2.1 and 2.2 we derive

$$\begin{aligned}
A(2) :=& \tfrac{4^{t-1}}{2}(-1)^{t-2}(t-2)\left[-(-1)^{t-2}\tfrac{2}{4^{t-1}}C_{t-2} - \tfrac{(-1)^{t-2}}{2\cdot 4^{t-2}}(t-1)C_{t-2} - \tfrac{(-1)^{t-1}}{4^{t-1}}tC_{t-1}\right]\\
&-4^{t-2}(-1)^{t-2}\left[(-1)^{t-2} - 2\tfrac{(-1)^{t-2}}{4^{t-2}}(t-1)C_{t-2}\right] - 4^{t-1}(-1)^{t-1}t\left[-(-1)^{t-1}\tfrac{2}{4^{t}}C_{t-1}\right.\\
&\left.-\tfrac{(-1)^{t-2}}{4^{t-2}}(t-1)C_{t-2}\left(-\tfrac{1}{8}\right) - \tfrac{(-1)^{t-1}}{4^{t-1}}tC_{t-1}\tfrac{1}{2} - \tfrac{(-1)^{t}}{4^{t}}(t+1)C_{t}\right]\\
&+\tfrac{4^{t-2}}{2}(-1)^{t-1}\left[(-1)^{t-1} - 2\tfrac{(-1)^{t-1}}{4^{t-1}}tC_{t-1} - \tfrac{(-1)^{t-2}}{4^{t-2}}(t-1)C_{t-2}\left(-\tfrac{1}{2}\right)\right]\\
&+\tfrac{4^{t-1}}{2}(-1)^{t-2}(t-1)\left[-(-1)^{t-2}\tfrac{2}{4^{t-1}}C_{t-2} - \tfrac{(-1)^{t-2}}{4^{t-2}}(t-1)C_{t-2}\tfrac{1}{2} - \tfrac{(-1)^{t-1}}{4^{t-1}}tC_{t-1}\right]\\
&-4^{t-2}(-1)^{t-2}\left[(-1)^{t-2} - 2\tfrac{(-1)^{t-2}}{4^{t-2}}(t-1)C_{t-2}\right]\\
=& -(t-2)C_{t-2} - (t-2)(t-1)C_{t-2} + \tfrac{t(t-2)}{2}C_{t-1} + 2(t-1)C_{t-2} + \tfrac{t}{2}C_{t-1}\\
&+\tfrac{t(t-1)}{2}C_{t-2} + \tfrac{t^2}{2}C_{t-1} - \tfrac{t(t+1)}{4}C_t - tC_{t-1} - 2(t-1)C_{t-2}\\
&-(t-1)^2C_{t-2} + \tfrac{t(t-1)}{2}C_{t-1} + 2(t-1)C_{t-2}\\
=& \left[-(t-2) - (t-2)(t-1) + 2(t-1) + \tfrac{t(t-1)}{2} - (t-1)^2\right]C_{t-2}\\
&+\left[\tfrac{t(t-2)}{2} + \tfrac{t}{2} + \tfrac{t^2}{2} - t + \tfrac{t(t-1)}{2}\right]C_{t-1} - \tfrac{t(t+1)}{4}C_t\\
=& \tfrac{-3t^2+11t-6}{2}C_{t-2} + \tfrac{t(3t-4)}{2}C_{t-1} - \tfrac{t(t+1)}{4}C_t\\
=& \tfrac{-3t^2+11t-6}{2}C_{t-2} + \tfrac{t(3t-4)}{2}\tfrac{2(2t-3)}{t}C_{t-2} - \tfrac{t(t+1)}{4}\tfrac{4(2t-1)(2t-3)}{t(t+1)}C_{t-2}\\
=& \tfrac{-3t^2+11t-6+4t^2-18t+18}{2}C_{t-2} = \tfrac{t^2-7t+12}{2}C_{t-2} = \tfrac{(t-3)(t-4)}{2}C_{t-2} = \tbinom{t-3}{2}C_{t-2}.
\end{aligned}$$

□

3.1 IR Conjecture

In this section we propose generalizations of the identities given in Theorem 3.1. The identities have been confirmed numerically in Maple (corresponding Maple Code is provided in APPENDIX section). To simplify the view of the convolution identities we introduce the following function: $F(n) = t - 2 - \sum_{i=1}^{n} s_i$, for a natural number n, where t is a natural number ($t \geq 3$).

Theorem 3.2. (***Theorem-Conjecture***) *For any positive integers t ($t \geq 3$) the following convolution identities take place*

$$\begin{aligned}
A(3) =& \sum_{s_1=1}^{t-3}\binom{s_1+1}{s_1}\binom{s_1-1}{2}C_{s_1}C_{F(1)}\\
&-\sum_{s_1=1}^{t-3}\binom{s_1+1}{s_1-1}\binom{s_1-1}{1}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}C_{F(2)}\\
&+\sum_{s_1=2}^{t-3}\binom{s_1+1}{s_1-2}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}\sum_{s_3=1}^{F(2)-1}C_{s_3}C_{F(3)} = \binom{t-3}{3}C_{t-2};
\end{aligned}$$

$$
\begin{aligned}
A(4)= & \sum_{s_1=1}^{t-3}\binom{s_1+1}{s_1}\binom{s_1-1}{3}C_{s_1}C_{F(1)}\\
& -\sum_{s_1=1}^{t-3}\binom{s_1+1}{s_1-1}\binom{s_1-1}{2}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}C_{F(2)}\\
& +\sum_{s_1=2}^{t-3}\binom{s_1+1}{s_1-2}\binom{s_1-1}{1}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}\sum_{s_3=1}^{F(2)-1}C_{s_3}C_{F(3)}\\
& -\sum_{s_1=3}^{t-3}\binom{s_1+1}{s_1-3}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}\sum_{s_3=1}^{F(2)-1}C_{s_3}\sum_{s_4=1}^{F(3)-1}C_{s_4}C_{F(4)}=\binom{t-3}{4}C_{t-2};
\end{aligned}
$$

$$
\begin{aligned}
A(5)= & \sum_{s_1=1}^{t-3}\binom{s_1+1}{s_1}\binom{s_1-1}{4}C_{s_1}C_{F(1)}\\
& -\sum_{s_1=1}^{t-3}\binom{s_1+1}{s_1-1}\binom{s_1-1}{3}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}C_{F(2)}\\
& +\sum_{s_1=2}^{t-3}\binom{s_1+1}{s_1-2}\binom{s_1-1}{2}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}\sum_{s_3=1}^{F(2)-1}C_{s_3}C_{F(3)}\\
& -\sum_{s_1=3}^{t-3}\binom{s_1+1}{s_1-3}\binom{s_1-1}{1}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}\sum_{s_3=1}^{F(2)-1}C_{s_3}\sum_{s_4=1}^{F(3)-1}C_{s_4}C_{F(4)}\\
& +\sum_{s_1=4}^{t-3}\binom{s_1+1}{s_1-4}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}\sum_{s_3=1}^{F(2)-1}C_{s_3}\sum_{s_4=1}^{F(3)-1}C_{s_4}\sum_{s_5=1}^{F(4)-1}C_{s_5}C_{F(5)}=\binom{t-3}{5}C_{t-2}.
\end{aligned}
$$

In general, for any positive integers t $(t\geq 3)$ and k $(1\leq k\leq t-3)$ the following convolution identities hold true

$$
\begin{aligned}
A(k)= & \sum_{s_1=1}^{t-3}\binom{s_1+1}{s_1}\binom{s_1-1}{k-1}C_{s_1}C_{F(1)}\\
& -\sum_{s_1=1}^{t-3}\binom{s_1+1}{s_1-1}\binom{s_1-1}{k-2}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}C_{F(2)}\\
& +\sum_{s_1=2}^{t-3}\binom{s_1+1}{s_1-2}\binom{s_1-1}{k-3}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}\sum_{s_3=1}^{F(2)-1}C_{s_3}C_{F(3)}\\
& -\sum_{s_1=3}^{t-3}\binom{s_1+1}{s_1-3}\binom{s_1-1}{k-4}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}\sum_{s_3=1}^{F(2)-1}C_{s_3}\sum_{s_4=1}^{F(4)-1}C_{s_4}C_{F(4)}\\
& +\ldots+(-1)^r\sum_{s_1=r}^{t-3}\binom{s_1+1}{s_1-r}\binom{s_1-1}{k-(r+1)}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}\cdots\sum_{s_{r+1}=1}^{F(r)-1}C_{s_{r+1}}C_{F(r+1)}\\
& +\ldots+(-1)^{k-2}\sum_{s_1=k-2}^{t-3}\binom{s_1+1}{s_1-(k-2)}\binom{s_1-1}{1}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}\cdots\sum_{s_{k-1}=1}^{F(k-2)-1}C_{s_{k-1}}C_{F(k-1)}\\
& +(-1)^{k-1}\sum_{s_1=k-1}^{t-3}\binom{s_1+1}{s_1-(k-1)}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}\cdots\sum_{s_{k-1}=1}^{F(k-2)-1}C_{k-1}\sum_{s_k=1}^{F(k-1)-1}C_{s_k}C_{F(k)}=\binom{t-3}{k}C_{t-2}.
\end{aligned}
$$

4 Conclusions

The authors believe that there is a well-favoured proofs of the identities exhibited in the paper. By publishing this paper we would like to draw an attention of the experts in combinatorics and expect more shorter proofs of the identities conjectured.

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Conflicts of Interest. The authors declare no conflict of interest.

A APPENDIX

In this section we provide a Maple code to verify the identities (as a sample it is chosen $t = 13$ and $k = 4$).

Maple Program:

Synopsis:

1. To verify the identities:
   - Choose t;
   - Generate Catalan numbers C_s, $s = 1, 2, ..., t-3$;
   - Choose k, where $k < t-3$;
   - Generate the identities.
2. Output is the confirmation.

Procedure:

> $t := 13;\ n := t-3;\ C_0 := 1;$

>

> **for** i **from** 1 **to** $n+1$ **do** $C_i := \dfrac{(2\cdot i)!}{(i+1)!\cdot i!}$ **end do**;

> $F(i) := t - 2 - \sum_{l=1}^{i} s[l];$

> $k := 4;$

> $ir\_identity :=$ **proc**(k, t)
>
> **local** $Z, G, i, j, r;$
>
> $F(i) := t - 2 - \sum_{l=1}^{i} s[l];$
>
> $Z[1] := \sum_{s[1]=1}^{t-3} \binom{s[1]+1}{s[1]} \binom{s[1]-1}{k-1} C_{s[1]} C_{F(1)};$
>
> **for** r **from** 2 **to** k **do**
>
> $G[r] := C[F(r)];$
>
> **for** j **from** 1 **to** $k-1$ **do**
>
> $G[r-j] := \sum_{s[r-j+1]=1}^{F(r-j)-1} C[s[r-j+1]]\cdot G[r-j+1];$
>
> **end do**;
>
> $Z[r] := (-1)^{r-1} \cdot \sum_{s[1]=r-1}^{t-3} \binom{s[1]+1}{s[1]-r+1} \binom{s[1]-1}{k-r} \cdot C[s[1]] \cdot G[1];$
>
> **end do**;
>
> **return** $\sum_{i=1}^{k} Z[i]$
>
> **end proc**:

> $ir\_identity(k, t) = \binom{t-3}{k} C_{t-2}$

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